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DYNAMIC RESPONSE OF LINEAR SYSTEMS TO MOVING STOCHASTIC SOURCES

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We present a generalized theory for dealing with the dynamic response of linear systems to moving sources. Stochastic characteristics of the response of linear systems to moving stochastic sources are theoretically analyzed based on the time-convolution expression established in this paper. We show that the random response of a linear system under a moving stationary stochastic source becomes a non-stationary process, for which the commonly used spectral analysis is not valid. To overcome this obstacle, the follow-up spectral analysis procedure is introduced. Statistical characteristics of the dynamic response are then given in the fixed and follow-up co-ordinates. A brief physical explanation related to time-frequency domain analysis is also provided. The theory developed in the paper can be universally applied to the moving source problem for linear systems.

1. INTRODUCTION

The current design procedures for transportation infrastructure (e.g., highway pavement, airport runaway, rail-track and bridge) are based on static theory. This is reasonable under the circumstance of low-speed vehicle motions. As the speed goes up, dynamic loads caused by vehicle vibration may result in significant differences between static and dynamic response of structures. In particular, it can become difficult to interpret the mechanism of certain failure phenomena, such as material fatigue, within the framework of static theory. In practice, these failures are related to the dynamic response of structures.

The dynamic response of continuum media such as beam, slab, and half-space under moving sources has been of interest for several decades. A number of studies have addressed this subject in various fields of physics, such as references [1, 2] for hydraulics, reference [3] for acoustics and references [4–10] for elastodynamics. One may refer to reference [11] for more detailed reviews. In most of these studies, however, only special cases are considered. For example, the most commonly used

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hypothesis in the literature is to assume that the moving source is a constant load with constant velocity. The formidable obstacle in the moving source problem is that the source position varies with time. This makes axisymmetric co-ordinates useless for representing the governing equations and also increases the complexity of the theoretical analysis due to the appearance of the velocity parameter. There is no generally applicable method for moving source problems.

There is evidence [11–14] indicating that vehicle-vibration-induced dynamic loads are moving stochastic loads that are primarily relevant to surface roughness of the structures. Assuming that other variable are constant, the higher the velocity of a vehicle, the stronger the vibration and the dynamic pavement loads [14, 15]. To determine the influence of roughness on structural response, we must regard the vibration-induced dynamic load as a moving stochastic source. A few investigations have addressed the dynamic response of media with the configuration of random moving load [16] and random foundation [17]. However, to the best knowledge of the authors, the study of the dynamic response of linear systems to moving stochastic sources has not been found in the literature.

The objective of this paper is to provide a theoretical foundation for the moving source problem. A time-convolution formulation is derived based on physical laws. This provides a general expression for the solution of the moving source problem. Deterministic and stochastic source conditions are both considered in the analysis. Aspects of random response including the mean, variance, correlation function, and power spectral density (PSD) are all considered in detail.

2. DESCRIPTION OF THE MOVING SOURCE PROBLEM

Consider the moving source problem described in reference [11]. A linear medium with region R and boundary B is initially at rest. The medium may be infinite or finite, such as a half-space or a multilayer medium. A moving source $F(\mathbf{x}, t)$ with a spatial amplitude distribution is applied on the plane $z = z_h$ of the medium. It then travels at constant speed v along a straight line (see Figure 1).

If the load is applied at the moment $t = 0$, it is called a suddenly applied moving load. Otherwise, the load is applied at the moment $t = -\infty$ and called a steadily applied moving load. The former corresponds to the transient response of the medium, and the latter corresponds to the steady state response of the medium. These two types of moving loads specified in reference [11] are respectively represented by

$$F(\mathbf{x}, t) = H[r_0^2 - (x - vt)^2 - y^2]\,\delta(z - z_h)p(t)H(t)/\pi r_0^2 \tag{1}$$

and

$$F(\mathbf{x}, t) = H[r_0^2 - (x - vt)^2 - y^2]\,\delta(z - z_h)p(t)/\pi r_0^2, \tag{2}$$

where $p(.)$ describes a source signature and r_0 is the radius of the source. In addition, $H(.)$ represents the Heaviside step function and $\delta(.)$ represents the Dirac

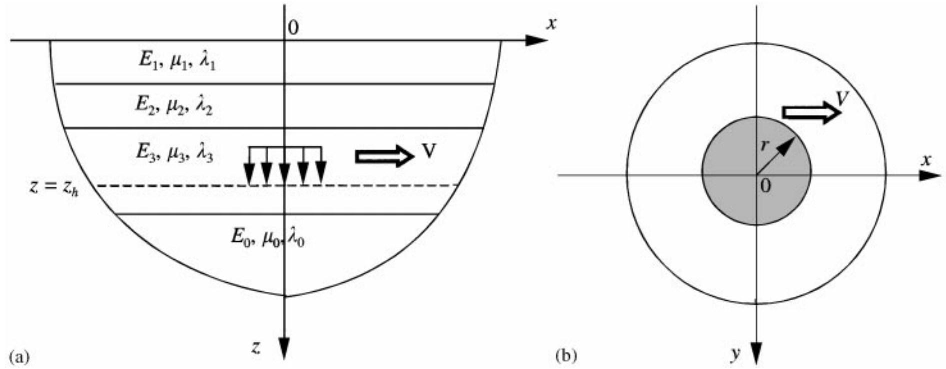

Figure 1. Sketch of a moving source: (a) side view, (b) top view.

delta function. These are defined respectively by

$$H(t - t_0) = \begin{cases} 0 & \text{if } t < t_0, \\ \frac{1}{2} & \text{if } t = t_0, \\ 1 & \text{if } t > t_0. \end{cases} \tag{3}$$

and

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\,\mathrm{d}x = f(x_0) \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x - x_0)\,\mathrm{d}x = 1. \tag{4}$$

The problem here is to compute the dynamic response of the linear medium under a moving source $F(\mathbf{x}, t)$.

3. DETERMINISTIC SOURCE

Initially, we consider the deterministic problem. That is, $p(t)$ presented in equation (1) and (2) is a deterministic function rather than a random process. It is convenient to assume a three-dimensional configuration with observation variable $\mathbf{x} = (x, y, z)$, source variable $\boldsymbol{\xi} = (\xi, \eta, \zeta)$, and time $t \geqslant 0$. Suppose a linear differential operator $\mathbf{O}$ describes the dynamic property of a physical system. Different systems possess different dynamic properties and operators. For instance, the operator of a specified elastodynamic problem is given by the well-known Navier–Stoke's field equations, Green's function is then defined as the fundamental solution of the system. In other words, for the problem discussed in the paper. Green's function corresponds to the solution of the system governing equations as the point source takes the form of a Dirac delta function in both spatial and temporal domain.

Without loss of generality, vanishing initial conditions are considered here. According to the causality for a realistic system, Green's function $\mathbf{G}(\mathbf{x}, t) = 0$

for $t < 0$. We may then write

$$\mathbf{O}[\mathbf{G}(\mathbf{x} - \xi, t - \tau)] = \delta(\mathbf{x} - \boldsymbol{\xi})\delta(t - \tau). \tag{5}$$

Here, the initiation of the source is delayed by τ. Therefore, the physical law requires that for Green's function, $t \leqslant \tau$. The properties of the medium under consideration are accounted for in the operator $\mathbf{O}$, while approximate interface conditions relate the field quantities in layers with different media. Equation (5) therefore applied to the classes of the configuration that are mentioned in the beginning of this paper. Since the steady-state solution can be derived from the transient solution, we initially need only to analyze the suddenly applied moving load.

We introduce a linear integral operator I as

$$I = \int_0^t \int_D f(\boldsymbol{\xi}, \tau) p(\tau)\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\tau, \tag{6}$$

where the kernel f describes a spatial amplitude distribution within finite domain D in co-ordinate $\boldsymbol{\xi} = (\xi, \eta, \zeta)$, and where the kernel p is a transient source signature with $p(t) = 0$ for $t \leqslant 0$. The response of a linear system to a finite source distribution with arbitrary signature is obtained by applying the integral operator to both sides of equation (5). Assuming that the operator is bounded, we may interchange the order of the integral operator and the $\mathbf{O}$. After carrying out the integration we find

$$\mathbf{u}(\mathbf{x}, t) = \int_0^t \int_D f(\boldsymbol{\xi}, \tau) \mathbf{G}(\mathbf{x} - \boldsymbol{\xi}, t - \tau)\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\tau, \tag{7}$$

where $\mathbf{u}(\mathbf{x}, t)$ represents the displacement field of the medium and the upper limit of the time convolution is determined by the time of interest.

If we set $f(\mathbf{x}, t) = F(\mathbf{x}, t)$, it is straightforward to see that the kernel $f(\mathbf{x}, t)$ can be replaced by equation (1). Substituting equation (1) into equation (7) we can rewrite $\mathbf{u}(\mathbf{x}, t)$ as

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t) &= \int_0^t p(\tau) \int_D \frac{H[r_0^2 - (\xi - vt)^2 - \eta^2]}{\pi r_0^2} \delta(\zeta - z_h) \mathbf{G}(\mathbf{x} - \xi, t - \tau)\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\tau \\ &= \int_0^t p(\tau) \int_D \frac{H(r_0^2 - \xi'^2 - \eta'^2)}{\pi r_0^2} \delta(\zeta') \mathbf{G}(x - \xi' - v\tau, y - \eta', z - \zeta' \\ &\qquad - \mathrm{z}_h, \mathrm{t} - \tau)\, \mathrm{d}\boldsymbol{\xi}'\, \mathrm{d}\tau \end{aligned} \tag{8}$$

using the transformation $\xi' = \xi - v\tau$, $\eta' = \eta$ and $\zeta' = \zeta$. The new finite domain D' corresponds to the spatial amplitude distribution in the new co-ordinate

$\boldsymbol{\xi}' = (\xi', \eta', \zeta')$. By defining the impulse unit response function $\mathbf{h}$ as

$$\mathbf{h}(\mathbf{x}, t) = \int_{D'} \frac{H(r_0^2 - \xi'^2 - \eta'^2)}{\pi r_0^2} \delta(\zeta') \mathbf{G}(x - \xi', t) \, \mathrm{d}\xi', \tag{9}$$

we can rewrite equation (8) as

$$\mathbf{u}(\mathbf{x}, t) = \int_0^t p(\tau) \mathbf{h}(x - v\tau, z - z_h, t - \tau) \, \mathrm{d}\tau. \tag{10a}$$

If the load is a steadily applied moving load, corresponding to the steady state response of the medium, the solution can be obtained by changing the lower limit of the integration in equation (10a) from 0 to $-\infty$, i.e.,

$$\mathbf{u}(\mathbf{x}, t) = \int_0^\infty p(t - \theta) \mathbf{h}(x - vt + v\theta, y, z, z - z_h, \theta) \, \mathrm{d}\theta, \tag{10b}$$

where $\theta = t - \tau$. It is also easy to see from equations (7) and (9) that the impulse unit response function $\mathbf{h}$ corresponds to the solution, since the applied load takes the form $F(\mathbf{x}, t) = H(r_0^2 - x^2 - y^2)\delta(z)\delta(t)/\pi r_0^2$.

Both equations (7) and (10a, b) show a universal representation for the solution of the moving source problem. In the case of a source from a surface, the analysis is similar, except that the solution is obtained by letting $z_h = 0$ in equations (10a, b). If the load is a constant with $p(t) = \bar{p}$, the transient response and the steady state response are given by

$$\mathbf{u}(\mathbf{x}, t) = \bar{p} \int_0^t \mathbf{h}(x - v\tau, y, z - z_h, t - \tau) \, \mathrm{d}\tau \tag{11}$$

and

$$\mathbf{u}(\mathbf{x}, t) = \bar{p} \int_{-\infty}^t \mathbf{h}(x - v\tau, y, z - z_h, t - \tau) \, \mathrm{d}\tau \tag{12}$$

respectively.

4. STOCHASTIC SOURCE CONDITION

A large class of time-dependent sources such as explosion and vehicle-induced dynamic loads may cause structural random response. The attributes of this response depend on both the characteristics (e.g., frequency content) and the structural properties (e.g., eigenfrequencies, damping, etc.) of the source. Generally, time-dependent loading shows statistical variations and, consequently, random response. Since time-dependent random variables are involved, the description of the response as deterministic is not sufficient. Thus, a stochastic process is needed to analyze a moving stochastic source.

Definition. A stochastic process $\chi(t)$ is called weakly stationary if its expectation is independent of time t and its correlation function depends only on the time interval

$\tau = t_2 - t_1$, i.e.,

$$\bar{\chi} = E[\chi(t)], \quad R_{\chi\chi}(\tau) = E[\chi(t_1)\chi(t_2)], \tag{13, 14}$$

where $\bar{\chi}$ and $R_{\chi\chi}$ are, respectively, mean and autocorrelation function of a stationary stochastic process [17].

It should be noted that in the case of Gaussian stochastic processes, weak stationary implies strong stationary [18]. According to Wiener–Khintchine theory, the PSD $S_{\chi\chi}(\omega)$ and correlation function $R_{\chi\chi}(\tau)$ of a stationary process constitute a pair of Fourier transforms,

$$S_{\chi\chi}(\omega) = (1/2\pi) \int_{-\infty}^{\infty} R_{\chi\chi}(\tau) \mathrm{e}^{-\mathrm{i}\omega\tau} \, \mathrm{d}\tau, \tag{15}$$

$$R_{\chi\chi}(\tau) = \int_{-\infty}^{\infty} S_{\chi\chi}(\omega) \mathrm{e}^{\mathrm{i}\omega\tau} \, \mathrm{d}\omega, \tag{16}$$

where ω is the angular frequency and τ the time interval.

In the derivation of equations (7) an (10a, b) we require no special assumptions on $p(t)$. Therefore, if $p(t)$ is a stochastic process, equations (10a, b) become integrals in the sense of Stieltjes integration [19]. Taking the expectation of both sides of equations (10a, b) and using the exchangeability of expectation and integration, we obtain the mean of the transient and the steady state response, i.e.,

$$E[\mathbf{u}(\mathbf{x}, t)] = \int_0^t E[p(\tau)] \, \mathbf{h}(x - v\tau, y, z - z_h, t - \tau) \, \mathrm{d}\tau, \tag{17}$$

$$E[\mathbf{u}(\mathbf{x}, t)] = \int_0^\infty E[p(t - \theta)] \, \mathbf{h}(x - v\tau + v\theta, y, z - z_h, \theta) \, \mathrm{d}\theta, \tag{18}$$

It is not difficult to obtain the spatial-time correlation functions for the transient and steady state response. That is,

$$R_{\mathrm{u}}(\mathbf{x}_1, \mathbf{x}_2; t_1, t_2) = \int_0^{t_2} \int_0^{t_1} R_p(\tau_2 - \tau_1) \mathbf{h}(x_1 - v\tau_1, y_1, z_1 - z_h, t_1 - \tau_1) \times \mathbf{h}(x_2 - v\tau_2, y_2, z_2 - z_h, t_2 - \tau_2) \, \mathrm{d}\tau_1 \, \mathrm{d}\tau_2, \tag{19}$$

$$R_{\mathrm{u}}(\mathbf{x}_1, \mathbf{x}_2; t_1, t_2) = \int_{-\infty}^{t_2} \int_{-\infty}^{t_1} R_p(\tau_2 - \tau_1) \mathbf{h}(x_1 - v\tau_1, y_1, z_1 - z_h, t_1 - \tau_1) \times \mathbf{h}(x_2 - v\tau_2, y_2, z_2 - z_h, t_2 - \tau_2) \, \mathrm{d}\tau_1 \, \mathrm{d}\tau_2, \tag{20}$$

where R_u and R_p are correlation functions of the displacement response $\mathbf{u}(\mathbf{x}, t)$ and the source $p(t)$ respectively. Letting $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}$, we obtain the time autocorrelation function

$$R_{\rm u}(\mathbf{x}; t_1, t_2) = \int_0^{t_2}\int_0^{t_1} R_p(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1)\mathbf{h}(x_1 - v\tau_1, y, z - z_h, t_1 - \tau_1)$$

$$\times\,\mathbf{h}(x - v\tau_2, y, z - z_h, t_2 - \tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2, \qquad (21)$$

and

$$R_{\rm u}(\mathbf{x}; t_1, t_2) = \int_0^{\infty}\int_0^{\infty} R_p(t_2 - t_1 + \theta_1 - \theta_2)\mathbf{h}(x - vt_1 + v\theta_1, y, z - z_h, \theta_1)$$

$$\times\,\mathbf{h}(x - v\tau_2 + v\theta_2, y, z - z_h, \theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2, \qquad (22)$$

where $\theta_j = t_j - \tau_j$ $(j = 1, 2)$. By substituting $t_1 = t_2 = t$ into equations (21) and (22), it is straightforward to find second moment functions, i.e., the mean-square functions of the random response.

One of the most significant properties of a linear system is that if the input of the system is a stationary process, the output is also a stationary process [19, 20]. However, this conclusion only applies to the fixed source problem (see reference [11]). To show this we introduce the following theorems.

Theorem 1. *Consider the steady state problem of a stochastic source with fixed position. If the random source (i.e., the input) to a linear system is a stationary stochastic process, then the random response (i.e., the output) is a stationary stochastic process.*

Proof. Since the random source is a stationary stochastic process, we have

$$E[p(t)] = \bar{p}, \qquad (23)$$

where $\bar{p}$ is the mean of the stationary stochastic process $p(t)$. Letting $v = 0$ and $E[p(t)] = \bar{p}$ in equations (17) and (18), for the source-fixed problem, we get the mean function of the transient and the steady state response,

$$E[\mathbf{u}(\mathbf{x}, t)] = \bar{p}\int_0^t \mathbf{h}(x, y, z - z_h, \theta)\,\mathrm{d}\theta, \quad E[(\mathbf{u}(\mathbf{x}, t)] = \bar{p}\int_0^{\infty} \mathbf{h}(x, y, z - z_h, \theta)\,\mathrm{d}\theta. \qquad (24, 25)$$

Letting $\theta_j = t_j - \tau_j$ $(j = 1, 2)$ in equation (21) and $v = 0$ in equations (21) and (22), we get the autocorrelation function

$$R_{\rm u}(\mathbf{x}; t_1, t_2) = \int_0^{t_2}\int_0^{t_1} R_p(t_2 - t_1 + \theta_1 - \theta_2)\mathbf{h}(x, y, z - z_h, \theta_1)\mathbf{h}(x, y, z - z_h, \theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \qquad (26)$$

for the transient solution and

$$R_{\mathrm{u}}(\mathbf{x}; t_1, t_2) = \int_0^\infty \int_0^\infty R_p(t_2 - t_1 + \theta_1 - \theta_2)\mathbf{h}(x, y, z - z_h, \theta_1)\mathbf{h}(x, y, z - z_h, \theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \tag{27}$$

for the steady state solution. Note that equation (25) is independent of t and that equation (27) depends only on the time difference $t_2 - t_1$. Therefore, the stochastic process is stationary and the proof is complete. □

Equations (24) and (26) are the general solutions for transient response. These are clearly not stationary because time t is contained in the upper limit of the integration as well as in the kernel function of the transient solution.

We now consider two deterministic problems. These are the problem with a suddenly applied moving source and the problem with steadily applied moving source. The sources for these two problems respectively take the form

$$F(\mathbf{x}, t) = H(r_0^2 - x^2 - y^2)\delta(z - z_h)\bar{p}H(t)/\pi r_0^2 \tag{28}$$

and

$$F(\mathbf{x}, t) = H(r_0^2 - x^2 - y^2)\delta(z - z_h)\bar{p}(t)/\pi r_0^2. \tag{29}$$

We can see that the right-hand sides of equations (24) and (25) correspond to the solutions of these deterministic problems. Furthermore, the steady state solution in equation (25) is time independent, therefore, it degrades to the static solution corresponding to a source with stationary position and amplitude, i.e., equation (29).

Theorem 2. *For the steady state problem with a moving stochastic source, the random response of a linear system is a non-stationary stochastic process even if the random signature $p(t)$ is a stationary stochastic process.*

Proof. The proof is based on the definition of stationary processes. In this case, the mean of the transient and the steady state response are respectively given by

$$E[\mathbf{u}(\mathbf{x}, t)] = \bar{p}\int_0^t \mathbf{h}(x - v\tau, y, z - z_h, t - \tau)\,\mathrm{d}\tau \tag{30}$$

and

$$E[\mathbf{u}(\mathbf{x}, t)] = \bar{p}\int_0^\infty \mathbf{h}(x - v\tau + v\theta, y, z - z_h, \theta)\,\mathrm{d}\theta. \tag{31}$$

Equations (21) and (22) give the autocorrelation function for the transient and the steady state response respectively. Since these functions depend on time, and not just the time lag, equations (15) and (16) cannot be satisfied. This shows that the steady state solution of a moving source problem cannot be a stationary stochastic process and the proof is complete. □

We again consider two deterministic problems. The sources for a suddenly applied moving load and a steadily applied moving load respectively take the forms of

$$F(\mathbf{x}, t) = H[r_0^2 - (x - vt)^2 - y^2]\,\delta(z - z_h)\bar{p}H(t)/\pi r_0^2 \tag{32}$$

and

$$F(\mathbf{x}, t) = H[r_0^2 - (x - vt)^2 - y^2]\,\delta(z - z_h)\bar{p}/\pi r_0^2. \tag{33}$$

Clearly, the right-hand side of equations (32) and (33) correspond to boundary conditions of the two deterministic problems.

For any physical system that is described by linear equations, Theorems 1 and 2 apply. It is useful to realize that although the system in Theorem 2 is a linear system, it essentially becomes a time-varying system when a moving source is applied. In contrast, the linear system in Theorem 1 is time invariant. Whether the source is fixed or moving, the transient response is always a non-stationary process because of the involved time variable. In the following sections, we will only discuss steady state solutions with regard to the moving stochastic source problem.

5. FREQUENCY DOMAIN ANALYSIS

Theorems 1 and 2 clearly indicate that there is an essential difference between the fixed source problem and the moving source problem. For the fixed source problem, the linear system is time invariant. So, we can directly obtain frequency components of the random source and the linear system by measuring the time history of random response at an arbitrary field point, say, $\mathbf{x}$, of the medium. In addition, spectral analysis (Fourier analysis) techniques for stationary processes are applicable to the random response. This is because the spectral analysis is used to represent a time-invariant variable in terms of the cumulative sum of a series of sinusoid signals that represent steady state functions of time. However, since the random response for the moving source problem is a non-stationary process, we cannot directly obtain frequency information by applying spectral analysis to sampled records at a fixed point.

Since transient solutions are time dependent rather than time-lag dependent, Fourier spectral analysis techniques based on equations (15) and (16) do not apply. Furthermore, equations (15) and (16) do not apply even to the steady state response of a linear system if the source is moving. Equations (15) and (16) are satisfied if and only if the source is a stochastic stationary process with fixed position.

We show that there are two ways to overcome this obstacle. Since the random response of the moving source problem is a non-stationary stochastic process, one way is to take advantage of spectral analysis of non-stationary processes. Essentially, this approach revises the definition of PSD of stationary processes. Another approach is to develop a special technique that is applicable to the moving source problem. In the following subsections, we will demonstrate these two approaches.

5.1. NON-STATIONARY STOCHASTIC PROCESS SPECTRAL ANALYSIS

Several kinds of PSD definitions have been developed for treating non-stationary process spectral analysis. Here we cite the generalized PSD developed in reference [21] to apply to the current problem. Applying the double Fourier transform to the time autocorrelation function $R_u(\mathbf{x}; \mathbf{t}_1, t_2)$, we get the generalized PSD,

$$S_u(\mathbf{x}; \omega_1, \omega_2) = (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_u(\mathbf{x}; t_1, t_2) \mathrm{e}^{-\mathrm{i}(\omega_2 t_2 - \omega_1 t_1)} \, \mathrm{d}t_1 \, \mathrm{d}t_2, \tag{34}$$

and the inverse transform gives the correlation function

$$R_u(\mathbf{x}; t_1, \mathrm{t}_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_u(\mathbf{x}; \omega_1, \omega_2) \mathrm{e}^{\mathrm{i}(\omega_2 t_2 - \omega_1 t_1)} \, \mathrm{d}\omega_1 \, \mathrm{d}\omega_2. \tag{35}$$

Here a sufficient condition for the existence of the generalized PSD is the absolute integrability of the form

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |R_u(\mathbf{x}; t_1, t_2)| \, \mathrm{d}t_1 \, \mathrm{d}t_2 < +\infty. \tag{36}$$

The impulse unit response function for the moving source problem depends on both the time t and the time interval τ. Consequently, the frequency response function for the moving source problem is a function of both frequency ω and time t. Here we denote the frequency response function as $\mathbf{H}(\mathbf{x}; \omega, t)$ and require that $\mathbf{h}(\theta) = 0$ for $\theta < 0$. We define $\mathbf{H}(\mathbf{x}; \omega, t)$ through

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{H}(\mathbf{x}; \omega, t) \mathrm{e}^{\mathrm{i}\omega t}. \tag{37}$$

By substituting the periodic excitation $\mathrm{e}^{\mathrm{i}\omega t}$ for $p(t)$ in equation (10b) and comparing with equation (37) we obtain frequency response for the impulse unit response function,

$$\mathbf{H}(\mathbf{x}; \omega, t) = \int_{-\infty}^{\infty} \mathbf{h}(x - vt + v\theta, y, z - z_h, \theta) \mathrm{e}^{\mathrm{i}\omega\theta} \, \mathrm{d}\theta. \tag{38}$$

Equations (38) and (31) yield the mean function in terms of the frequency response function,

$$E[\mathbf{u}(\mathbf{x}, t)] = \bar{p} \mathbf{H}(\mathbf{x}; 0, t), \tag{39}$$

where $\bar{p} = E[p(t)]$. For the time autocorrelation function, we have

$$R_u(\mathbf{x}; t_1, t_2) = \int_{-\infty}^{\infty} \mathbf{H}(\mathbf{x}; \omega, t_1) S_p(\omega) \mathbf{H}(\mathbf{x}; -\omega, t_2) \mathrm{e}^{\mathrm{i}\omega_2(t_2 - t_1)} \, \mathrm{d}\omega, \tag{40}$$

where $S_p(\omega)$ is the load PSD. Consequently, the mean-square function is

$$\psi_{\rm u}^2(\mathbf{x}; t) = \int_{-\infty}^{\infty} |\mathbf{H}(\mathbf{x}; \omega, t)|^2 S_p(\omega)\,\mathrm{d}\omega \tag{41}$$

and the generalized PSD for the random response of the system is

$$S_{\rm u}(\mathbf{x}; \omega_1, \omega_2; t_1, t_2) = \mathbf{H}(\mathbf{x}; \omega_1, t_1) S_p(\omega) \mathbf{H}(\mathbf{x}; -\omega_2\ t_2), \tag{42}$$

where we have used the relation

$$S(\omega_1, \omega_2) = \delta(\omega_1 - \omega_2) S(\omega_1) \tag{43}$$

in which $S(\omega_1)$ is the PSD of a stationary process.

Although the response-excitation relationship of a linear system can be established within the framework of non-stationary process spectral analysis, it is difficult to provide a physical explanation for the generalized PSD. Also, we cannot obtain the generalized PSD from only a small sample of data. These shortcomings led us to develop a follow-up spectral analysis technique, by which the commonly used spectral analysis technique is still applicable [11].

5.2. FOLLOW-UP SPECTRAL ANALYSIS

Using equation (10b) we obtain the following expression if we consider random response of a moving field point $(x + vt, y, z)$:

$$\mathbf{u}(x + vt, y, z, t) = \int_0^{\infty} p(t - \theta)\mathbf{h}(x + v\theta, y, z - z_h, \theta)\,\mathrm{d}\theta. \tag{44}$$

Clearly, the moving field point $(x + vt, y, z)$ is travelling at the same speed as the moving source. Suppose a new co-ordinate $\mathbf{X}$ exactly follows the moving source. The relation of this follow-up coordinate (i.e., moving co-ordinate) and the old co-ordinate $\mathbf{X}$ (i.e., fixed co-ordinate) is $x' = x - vt$, $y' = y$, $z' = z$. The moving field point $(x + vt, y, z)$ in the fixed co-ordinate becomes a stationary field point in the follow-up co-ordinate; thus from equation (44) we obtain

$$\mathbf{u}(\mathbf{x}', t) = \int_0^{\infty} p(t - \theta)\mathbf{h}(x + v\theta, y, z - z_h, \theta)\,\mathrm{d}\theta. \tag{45}$$

Taking the expectation of both sides of equation (45), we immediately identify the mean of the random response at field point $\mathbf{x}'$ in the follow-up co-ordinate,

$$E[\mathbf{u}(\mathbf{x}', t)] = \bar{p} \int_0^{\infty} \mathbf{h}(x + v\theta, y, z - z_h, \theta)\,\mathrm{d}\theta, \tag{46}$$

where $E[p(t)] = \bar{p}$. Similarly, the time autocorrelation function for the random response at field point $\mathbf{x}'$ in the follow-up coordinate becomes

$$R_{\mathrm{u}}(\mathbf{x}'; \tau|_{\tau=t_2-t_1}) = \int_0^\infty \int_0^\infty R_p(\tau + \theta_1 - \theta_2)\mathbf{h}(x + v\theta_1, y, z - z_h, \theta_1) \times \mathbf{h}(x + v\theta_2, y, z - z_h, \theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2. \quad (47)$$

From equations (46) and (47) it is clear that in the follow-up co-ordinates the random response at field point $\mathbf{x}'$ has been converted to a stationary process. From equation (31) we see that the mean function of a linear system in the follow-up co-ordinate responding to a moving stationary stochastic source is equivalent to that of the same system in the fixed co-ordinate system. However, the fixed co-ordinate system is responding to a moving deterministic source whose amplitude is a constant equal to the mean of the moving stationary stochastic source.

Letting $t_1 = t_2 = t$, we write the mean-square function for the random response in the follow-up co-ordinate as

$$\psi_{\mathrm{u}}^2(\mathbf{x}', t) = \int_0^\infty \int_0^\infty \psi_p^2(\theta_2 - \theta_1)\mathbf{h}(x + v\theta_1, y, z - z_h, \theta_1)\mathbf{h}(x + v\theta_2, y, z - z_h, \theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2. \quad (48)$$

In addition, if $p(t)$ is assumed to be a stationary process with zero mean, we have

$$\sigma_p^2 = Var[p(t)] = \psi_p^2 = R_p(\tau|_{\tau=0}) = \text{constant}. \quad (49)$$

where σ_p and $Var[p(t)]$ are, respectively, the standard deviation and variance of $p(t)$. So we can rewrite equation (48) as

$$\sigma_u^2(\mathbf{x}', t) = \psi_u^2(\mathbf{x}', t) = \left[\int_0^\infty \mathbf{h}(x + v\theta, y, z - z_h, \theta)\,\mathrm{d}\theta\right]^2 R_p(\tau|_{\tau=0}), \quad (50)$$

where σ_{u} and ψ_{u}^2 are the standard deviation and variance of the displacement field respectively.

The physical explanations of equations (46)–(48) are essentially different from the usual explanations of random problem in fixed co-ordinates, such as equations (25) and (27). The latter refer to the time average and time autocorrelation of the random response at a fixed point $\mathbf{x} = (x, y, z)$, whereas the former refer to the spatial average and spatial-time correlation for the random response at a moving field $(x + vt, y, z)$ in fixed co-ordinates or a stationary field point $\mathbf{x}' = (x', y', z')$ in the follow-up co-ordinates. Figures 2 and 3 illustrate the

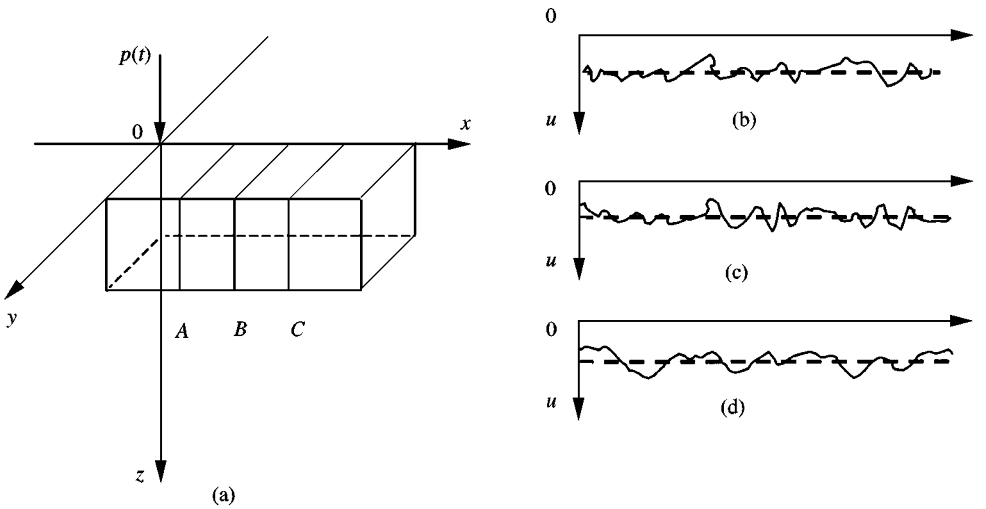

Figure 2. Random response of field points in fixed coordinates: (a) location of points $A(x, y, z)$, $B(x + x_0, y, z)$, $C(x + zx_0, y, z)$ and random responses of (b) A, (c) B and (d) C; $v = 0$.

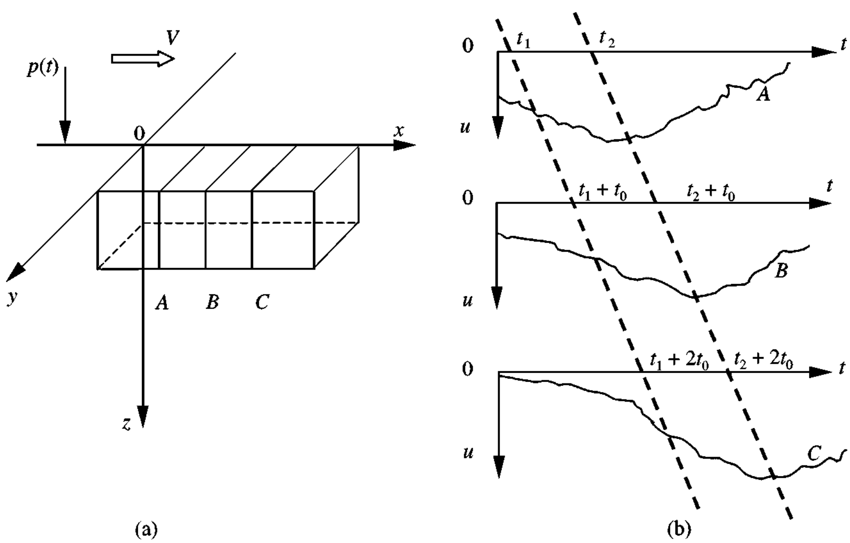

Figure 3. Random response of field points in follow-up coordinates: (a) field point location as in Figure 2, (b) responses of field points; $t_0 = x_0/v$.

random response of a fixed point in fixed co-ordinates and in the follow-up co-ordinates respectively.

We define the relationship between the frequency response function and the impulse unit response function in the follow-up coordinates as

$$\mathbf{H}(\mathbf{x}', \omega, t) = \int_0^\infty \mathbf{h}(x + v\theta, y, z - z_h, \theta)\mathrm{e}^{-\mathrm{i}\omega\theta}\,\mathrm{d}\theta. \tag{51}$$

Taking the Fourier transform of equation (47), the right-hand side becomes the follow-up PSD in follow-up co-ordinates. From equation (51), the left-hand side of equation (47) leads to an expression of the follow-up PSD in relation to the load PSD, i.e.,

$$S_{\mathrm{u}}(\mathbf{x}'; \omega, v) = |\mathbf{H}(\mathbf{x}'; \omega, v)|^2 S_p(\omega). \tag{52}$$

Similarly, an expression for the time autocorrelation function can be obtained by taking the Fourier inverse transform of equation (52), i.e.,

$$R_{\mathrm{u}}(\mathbf{x}; \tau|_{\tau = t_2 - t_1}) = (2\pi)^{-1} \int_{-\infty}^{\infty} |\mathbf{H}(\mathbf{x}'; \omega, v)|^2 S_p(\omega) \mathrm{e}^{\mathrm{i}\omega\tau} \, \mathrm{d}\omega. \tag{53}$$

Hence, the mean-square function is described by

$$\psi_{\mathrm{u}}^2(\mathbf{x}') = R_{\mathrm{u}}(\mathbf{x}'; \tau|_{\tau=0}) = (2\pi)^{-1} \int_{-\infty}^{\infty} |\mathbf{H}(\mathbf{x}'; \omega, v)|^2 S_p(\omega) \, \mathrm{d}\omega. \tag{54}$$

Generally, the response PSD at a fixed point $\mathbf{x} = (x, y, z)$ in a fixed co-ordinate represents the energy distribution in different frequency ranges. This means that we may conduct a stationary process spectral analysis of the time history of the random response at field point A (or B or C, etc.; see Figure 2). However, the follow-up PSD shown on the left-hand side of equation (52) indicates the frequency component of the random response at a fixed point $\mathbf{x}' = (x', y', z')$ in the follow-up co-ordinate or a series of moving field points in the fixed co-ordinate. This implies that stationary process spectral analysis can be performed on the random response at a field point on the line AC for any given time t_1 (see Figure 3).

6. DISCUSSION

According to the theory of linear partial differential equations, we may construct solutions to equations by integrating the fundamental solution or the so-called Green's function of the equation. This can be done because of the superposition principles for linear equations [22].

Response information for both the amplitude distribution and the frequency component of media to moving sources is needed for optimum control and performance prediction of structures. To fulfill this purpose, it is necessary to use spectral analysis techniques to obtain the required information. Since currently available techniques for non-stationary process spectral analysis are not necessarily valid in practice, the follow-up spectral analysis developed in the paper provides a sound theoretical basis and powerful technique. The advantage to using follow-up spectral analysis is that the physical explanations of the follow-up PSD allow us to interpret the results of the moving source problem just like the results of the source-fixed problem. Therefore, we recommend follow-up PSD rather than generalized PSD (non-stationary process spectral analysis) as a more useful technique for both theoretical analysis and practical applications in the future.

Readers may refer to reference [11] for more detailed application of the theory of this paper.

7. CONCLUSION

The response of linear systems to moving stochastic sources is analyzed in the paper. We show that although the random response of a linear system under a fixed stationary stochastic source retains the property of stationary, the same is not true under a moving stationary stochastic source. Follow-up co-ordinates are developed to overcome the difficulty of performing spectral analysis for non-stationary processes. This paper presents a theoretical foundation for treating the moving deterministic source problem and the moving stochastic source problem. Conclusions drawn here are applicable to linear systems in the fields of elastodynamics, acoustics, etc.

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APPENDIX A: NOTATION

$\mathbf{F}(\mathbf{x}, t)$	moving source
$H(\cdot)$	Heaviside step function
$\delta(\cdot)$	Dirac delta function
v	source velocity
$p(t)$	source magnitude
$\mathbf{G}[\cdot]$	Green's function
$\mathbf{u}[\cdot]$	response function of linear systems
$\mathbf{h}[\cdot]$	impulse response function
$\mathbf{H}[\cdot]$	frequency response function
$S(\cdot)$	power spectral density
$R(\cdot)$	correlation function
$E[\cdot]$	expectation
σ^2	variance
ψ	standard deviation